Brief Mathematical Analysis on Topics of General and Special Relativity

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Abstract. This article was written to graduate and postgraduate students of physics. We intend to be didactical using a mathematical approach in Riemannian spaces as rigorous as possible. Brief mathematical analysis are done on some topics in General (GR) and Special Relativity (SR).
Key words: General Relativity; Riemannian Geometry; Special Relativity.

(I) Introduction.

Are briefly analyzed only mathematical aspects of some topics in Special Relativity and General Relativity. Comparison with experimental results can be seen in references mentioned in the text. In **Section 1** we see **Manifolds**: coordinate transformations, metric and affine spaces. In **Section 2** are given examples of metric spaces and their geometries. In **Section 3** we define Geodesics. In **Section4** are analyzed local Cartesian or local inertial coordinate-systems. In **Section 5** are shown effects due to the Earth gravitational field (*Schwarzschild Metric*). In **Section 6** we see the Equivalence and the Geodesic Principles. In **Section 7** we consider accelerated coordinate-systems in SR. In **Section 8** we calculate spatial distances and time intervals in GR.

(1) Manifolds.

In elementary analytical geometry[1-3] the fundamental notion is that geometrical points are defined by means of its coordinates. Thus, in the geometry of the Euclidean plane, for instance, a point is specified by giving its two Cartesian coordinates (X,Y) or its polar coordinates (r,θ). All points which together constitute the plane are said to form a *2-dimensional manifold* or *2-dimensional space* of points, the number of dimensions of the manifold being equal to the number of independent coordinates required to specify a point in it. Generalizing this idea, an *n-dimensional manifold* or *n-dimensional space* of points is one for which n independent real numbers $(x^1,x^2,...,x^n)$ are required to specify every point completely. These n numbers are denoted collectively by $\{x^i\}_{i=1,..,n}$ and are called the *coordinates* of the point. For the moment the manifolds are assumed to have no *structure* except that it is *continuous* in the sense that, in the neighborhood of every point (x) there are other points whose coordinates differ infinitesimally from those of (x). Such neighboring point has coordinates (x + dx), the small quantities $(dx^1, dx^2,...,dx^n)$ being called the differentials of the coordinates (x). Being these coordinates independent we have $\partial x^\nu/\partial x^\mu = \delta_\mu{}^\nu$.

(1.1)Coordinate transformations.

Let us consider two manifolds $X =\{x^i\}_{i=1...n}$ and $X'=\{x'^j\}_{j=1...m}$ .To perform the coordinates transformation $X \rightarrow X'$ it is necessary that the manifolds have the same dimensions, that is, n = m. The operation by which the coordinates (x) of every point in the manifold are altered to (x') is called *coordinate transformation*. The coordinate transformation $X \rightarrow X'$ are defined by n equations $x'^\lambda = f^\lambda(x^1, x^2,,...,x^n)$ where $\lambda = 1,2,3,...,n$ and the functions f^λ are soluble in such a way that would be possible to obtain the inverse transformation, $X' \rightarrow X$, that is, $x^\lambda = g^\lambda(x'^1, x'^2,,...,x'^n)$.[1-3] Differentiating partially the $x'^\lambda(\partial x^\nu)$ we obtain the transformation law for the differentials

$$dx'^\lambda = (\partial f^\lambda/\partial x_\nu)dx^\nu = (\partial x'^\lambda/\partial x_\nu)dx^\nu \qquad (1.1.1).$$

Mathematical conditions that this transformation be (theoretically) possible is that the determinant of the coefficients (*Jacobian*) must not vanish

$$\det\left\|\partial x'^{\lambda}/\partial x_{\nu}\right\| \neq 0 \qquad (1.1.2)$$

at all points, or nearly all points. If this inequality does not hold, then the equations $X'=\{x'^{j}\}_{j=1...m}$ do not a legitimate this set of coordinates.[4]

(1.2) Space Affine: Covariant and Contravariant Tensors and Scalars.

Any aggregate of n quantities U^{μ} which transform from S to S´ like

$$U'^{\nu} = (\partial x'^{\nu}/\partial x_{\mu})\, U^{\mu} \qquad (1.2.3)$$

is called a *contravariant vector* (*1st order tensor*). Quantities V_{μ} which transform like

$$V'_{\nu} = (\partial x^{\mu}/\partial x'_{\nu})\, U_{\nu} \qquad (1.2.4)$$

are called *covariant vector.* Although covariant and contravariant vectors have nothing to do with each other their *inner product* defined as $V_{\mu}\, U^{\mu}$ is an *invariant*, that is, it is independent of the coordinate system. Indeed,

$$V'_{\mu}\, U'^{\mu} = (\partial x^{\alpha}/\partial x'^{\mu})\, (\partial x'^{\mu}/\partial x^{\beta}) V_{\alpha}\, U^{\beta} = (\partial x^{\alpha}/\partial x^{\beta})\, V_{\alpha}\, U^{\beta} = \delta_{\beta}{}^{\delta}\, V_{\alpha}\, U^{\beta},$$

that is,

$$V'_{\mu}\, U'^{\mu} = V_{\alpha}\, U^{\alpha} = \text{Invariant} \qquad (1.2.5),$$

showing that their *inner product* is a *scalar.*

Consider a function $F(x^{\mu}) = F(x'^{\nu})$. Since $dF(x^{\mu}) = dF(x'^{\nu})$ the differential dF is an *invariant*, that is, it is a *scalar*.

A space in which covariant and contravariant vectors exist separately is called *affine*. In other words, in an affine space a *vector* does not exist; but covariant and contravariant vectors exist independently of each other Such spaces are very general and have been extensively studied by mathematicians.[5] The *scalar* is *0th order tensor* and U_{ν} or U^{ν} are *first order tensors*. More details about definitions and properties of covariant and contravariant tensors of any order can be seen elsewhere.[1-5] For instance, a quantity $U_{\mu\nu}$ behaves like a 2.nd order covariant tensor if it transforms as

$$U'_{\mu\nu} = (\partial x^{\alpha}/\partial x'^{\mu})\, (\partial x^{\beta}/\partial x'^{\nu})\, U_{\alpha\beta} \qquad (1.2.6).$$

Higher order tensor can be seen, for instance, in references 1-5. In GR there are two important tensor, one of rank four $R^{\sigma}{}_{\lambda\nu\mu}$ called *Riemann-*

Christoffel tensor and one of rank two $R_{\lambda\mu}$ named *Ricci* tensor. These are shown explicitly in Appendix A.

(1.3) Metric Spaces.

For the moment the manifolds were assumed to have no *structure* except that it is continuous in the sense that, in the neighborhood of every point (x) there are other points whose coordinates differ infinitesimally from those of (x). In particular no definition of distance ds between pair of points has been given. In arriving at such a definition is introduced a *metric* that contains within itself the essentials of the *geometry* of the space.

In fundamental physical theories we deal with objects or quantities which are independent of the particular choice of the mode of description. In physics it is desirable to deal with a space in which the concept of *distance* and *tensors* exist as an objective physical reality. Such spaces are called *metric*. Before to define rigorously *metric spaces* let us remember that in physics the basic observable geometric quantity is the distance ds between two points. For instance, the most familiar metric space is 3-dimensional Euclidean space. In fact, a "metric" is the generalization of the Euclidean metric arising from the four long-known properties of the Euclidean distance. In a 3-dim Euclidean space, in Cartesian coordinates, $(x^1, x^2, x^3) = (x, y, z)$ the infinitesimal distance ds between two points is defined by the invariant $ds^2 = dx_\mu dx^\mu$, that is

$$ds^2 = dx^2 + dy^2 + dz^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{ij}\, dx^i dx^j, \qquad (1.3.1)$$

where $g^{ij} = \delta_{ij}$ is called "*metric tensor*". The distance ds between the same points must equal to that given by (1.3.1) independently of the coordinate system adopted. For instance, in spherical coordinates $x'^1 = r$, $x'^2 = \theta$ and $x'^3 = \varphi$ we must have [1-5]

$$ds^2 = ds'^2 = r^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2 = g'_{ij}\, dx'^i dx'^j \qquad (1.3.2),$$

where now the metric tensor is given by $g'_{ij} = (1, r^2, r^2\sin^2\theta)$.

Note that to get the above results the transformation $X \rightarrow X'$ is given by[1-5] $x^1 = r \sin\theta \cos\varphi$, $x^2 = r \sin\theta \sin\varphi$, $x^3 = r \cos\theta$. The distance ds between two points *must be independent* of the coordinates system. Similarly, vectors like velocity **V**, force **F**, distance **r**, ...must be independent of the coordinates used to represent them. Let us see how these ideas can be generalized involving n-dimensional manifolds. Before

to do this note that ds^2 in the 4-dim in the SR pseudo- Euclidean Minkowski space $x^\lambda = (x^1,x^2,x^3,x^4 = x^o= ct)$ is written in different ways in text books and papers. That is, we have $ds^2 = (x^1)+(x^2)^2+(x^3)^2- c^2dt^2$ where $g_{11} = g_{22} = g_{33} = 1$, $g_{ik} = 0$ $(i\neq k)$ and $g_{oo} = -1$ or $- ds^2$ and, sometimes, $ds^2 = c^2 dt^2 -\{(x^1)+(x^2)^2+(x^3)^2\}$ with $g_{11} = g_{22} = g_{33} = -1$ $g_{ik} = 0$ $(i\neq k)$ and $g_{oo} = 1$. Since ds^2 is invariant this equation is valid in all reference system. An interesting case is a clock at rest in a reference S_o .Since it is at rest its displacements vanish, $dx_o = dy_o = dz_o = 0$. Let us denote the time measured by this clock by $dt = d\tau$, where the time τ measured in S_o is called **proper time**, so $ds^2= - c^2 d\tau^2$. Now, let us consider a system S which moves with constant velocity v relatively to S_o . An observer in S will observe that this clock in a time interval dt suffer displacements dx, dy and dz in S. So, we can put $ds^2= - c^2 d\tau^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$. In this way, taking into account that $v^2 = (dx^2 + dy^2 + dz^2)/dt^2$, we obtain $d\tau = [1 - (v/c)^2]^{1/2} dt$, as seen in basic SR courses.[6]

There are spaces in which covariant and contravariant vectors do not exist independently but they can be converted into each other. These have a further property that the index of a contravariant vector U^μ can be lowered to become U_μ by the operation

$$U_\mu = g_{\mu\nu} U^\nu \qquad (1.3.3).$$

The reverse transformation is would be given by

$$U^\nu= g^{\nu\mu} U_\mu \qquad (1.3.4),$$

thus, $U_\mu = g_{\mu\nu} U^\nu = g_{\mu\nu} g^{\nu\alpha} U_\alpha$ which means that $g_{\mu\nu} g^{\nu\alpha} = \delta_\mu{}^\alpha$. In other words, $g_{\mu\nu}$ is the inverse of $g^{\mu\nu}$, and vice-versa, that is, $g^{\mu\nu} = M^{\mu\nu}/|g|$, where $|g|$ is the determinant of $g_{\mu\nu}$ and $M^{\mu\nu}$ is the minor of the element $g_{\mu\nu}$. In a metric space a *vector* is represented by **U** and has an objective meaning, independently of the adopted mathematical representation. Identical properties are obeyed for higher order tensors,covariant or contravariant[1-5] The *square* of a vector $\mathbf{U}^2$ is an *invariant* defined by the *inner product*

$$\mathbf{U}^2 = U_\mu U^\mu = g_{\mu\nu} U^\mu U^\nu = g^{\alpha\beta} U_\alpha U_\beta \qquad (1.3.5).$$

Similarly, we define the square $d\mathbf{s}^2 = ds^2$ of the differential dx^μ by:

$$ds^2 = d\mathbf{s}^2 = g_{\mu\nu} dx^\mu dx^\nu \qquad (1.3.6).$$

The quantity ds (1.3.6) is called the *line element* of the metric space. It expresses the invariance of ds^2, also named *fundamental quadratic form* of metric spaces. An important assumption which is not explicitly expressed in usual textbooks, is that the *basic observable of the metric geometry is the distance ds*.[4] Among other things, we can deduce from (1.3.5) that $g_{\mu\nu}$ is a *symmetric tensor*, that is, $g_{\mu\nu} = g_{\nu\mu}$. For instance, Riemannian spaces used in General Relativity (GR) are metric. In most of the applications of these Riemannian spaces, special types of spaces (said to be diagonal) are considered in which $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

It can be proved that the functions $g_{\mu\nu}$ are components of a covariant tensor of rank two. Indeed if in the coordinate transformation $X \rightarrow X'$:

$$ds^2 = g'_{\lambda\mu}\, dx'^{\lambda} dx'^{\mu} = g_{\lambda\mu}\, dx^{\lambda} dx^{\mu} ,$$

using (1.2.6), that is, taking $dx'^{\lambda} = (\partial x'^{\lambda}/\partial x_{\mu})dx^{\mu}$ we get

$$g'_{\alpha\beta}(x') = g_{\lambda\mu}(x')(\partial x^{\lambda}/\partial x'^{\alpha})(\partial x^{\mu}/\partial x'^{\beta}), \qquad (1.3.7),$$

where $x^i = x^i(\{x'^j\})$ showing that $g_{\mu\nu}$ transforms as a 2.nd order covariant tensor according to (1.2.6). Vice-versa we must have when $X' \rightarrow X$:

$$g_{\alpha\beta} = g_{\mu\nu}(\{x^j\})\,(\partial x'^{\mu}/\partial x_{\alpha})\,(\partial x'^{\nu}/\partial x_{\beta}) \qquad (1.3.8).$$

Remembering that the space-time metric is defined by the line invariants[1-3] $ds^2 = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}$, $g_{\mu\nu}$ e x^{λ} are, respectively, the metric tensor and the coordinates of the referential system.

Note that in all general "real" space-times the determinant |g| of the g_{ik} must be $|g| < 0$.[1,3] The gravitational field is named *constant* when all components of the tensor $g_{\alpha\beta}$ do not depend of the temporal coordinate x_o. When the components $g_{o\beta} = 0$ the champ is said to be *static*. It is *stationary* when the components $g_{o\beta} \neq 0$.

(2) Examples of Metric Spaces and Geometries.

Fundamental aspects that are very important to be mentioned are those given by relations between the line elements ds. By coordinate transformations different geometries be generated.[2,3]

(2.1) Two and Three-dim Euclidean spaces.[3]

In a 2-dim Cartesian orthogonal coordinates ($X = x^1$,$Y= x^2$) the metric ds is given by, $\quad ds^2 = dX^2 + dY^2 \qquad (2.1.1).$

This formula, where $g^{11} = g^{22} = 1$, that expresses the metric of the space defined by the variables (X,Y) contains the essentials of the *Plane Euclidean Geometry*: straight line is the shortest distance between two points, parallel lines are of infinite length and do not intersect in any finite part of the plane, and so on... These properties that are due to the use of Cartesian coordinates disappear if other types of coordinate-systems are employed. For example, if we transform from the Cartesian coordinates $X = x^1$ and $Y= x^2$ to polar coordinates $r = x'^1$, $\theta = x'^2$ by the equations $x^1= x'^1 \cos x'^2$ and $x^2 = x'^1 \sin x'^2$ the metric (2.1.1) becomes

$$ds^2 = dr^2 + r^2 d\theta^2 \qquad (2.1.2),$$

with $g^{11} = 1$ e $g^{22} = r^2$. It *does not describe* a Plane Geometry.

In 3-dim Euclidean space, when Cartesian coordinates (X,Y,Z) are used, the metric has the form

$$ds^2 = dX^2 + dY^2 + dZ^2 \qquad (2.1.3),$$

which is again the statement of Pythagora´s theorem. If these coordinates are changed as $(x^1= X, x^2 = Y, x^3 = Z) \rightarrow (x'^1= r, x'^2 = \theta, x'^3 = \varphi)$ according to $x^1 = x'^1 \sin x'^2 \cos x'^3$, $x^2 = x'^2 \sin x'^2 \sin x'^3$ and $x^3 = x'^1 \cos x'^2$ we see that (2.1,3) becomes written as

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \qquad (2.1.4).$$

Taking r = constant = a the metric becomes (with $g_{11} = a^2$ and $g_{22} = a^2 \sin^2\theta$)

$$ds^2 = a^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \qquad (2.1.5),$$

that describes the geometry of a 2-dim surface of a sphere with radius r = a which is intrinsically different from that of the 2-dim Euclidean plane: straight lines are replaced by great circles with finite length, and necessarily intersect, with no parallels in the Euclidean sense, and so on...

Note that these two sets of equations in 2 and 3 dimensions have the property of being soluble for the (x´) in terms of the (x); for example, 2-dim case the equations we have the relations

$$x'^1 = \{(x^1)^2 + (x^2)^2\}^{1/2} \quad \text{and} \quad x'^2 = \tan^{-1} (x^2/x^1).$$

In the 3-dim case we have similar relations, but more complicated.

(2.2) 2-dim Coordinate-System (u,v) where X = uv and Y = $(u^2+v^2)/2$.

In this case $ds^2 = dX^2 + dY^2$ becomes written as

$$ds^2 = (u^2 + v^2)du^2 + 4uvdudv + (u^2+v^2)dv^2, \qquad (2.2.1),$$

with $g_{11} = u^2 + v^2$, $g_{12} = g_{21} = 4u$ and $g_{22} = g_{11} = u^2 + v^2$, respectively, where the orthogonality property is masked in the new coordinate-system. What kind of geometry does this (u,v) coordinates describe? This question is very difficult to be answered in a general case.

(2.3) 3-dim manifold (X,Y,Z) with the metric

$$ds^2 = dX^2 + dY^2 - dZ^2 \qquad (2.3.1),$$

($g_{11} = g_{22} = 1$ and $g_{33} = -1$) is a 3-dim Minkowski space different from the Euclidean one. If the coordinates are transformed to (r_1,θ,φ) like

$$X = r_1 \sinh\theta \cos\varphi\ ,\ \ Y = r_1 \sinh\theta \sin\varphi \quad e \quad Z = r_1 \cosh\theta$$

we verify that (2.3.1) becomes

$$ds^2 = - dr_1^2 + r_1^2\, d\theta^2 + r_1^2 \sinh^2\theta\, d\varphi^2 \qquad (2.3.2).$$

Taking r = a = constant we obtain $ds^2 = a^2 (d\theta^2 + r_1^2 \sinh^2\theta\, d\varphi^2)$. With a new coordinate transformation $\theta \rightarrow r$, defined by $\sinh\theta = r/(1- r^2/4)$ we get

$$ds^2 = a^2 (dr^2 + r^2 d\varphi^2)/(1- r^2/4)^2 \qquad (2.3.3).$$

From (2.3.3), r = 0 when $\theta = 0$ and r = 2 when $\theta \rightarrow \infty$. Again, it is difficult to determine what kind of geometry is defined by (2.3.1)...

(3)Geodesics. [3]

In preceding sections have been performed changes of tensors when the coordinates of the points of a Riemannian space are transformed. Now we consider changes of a different kind that arise when the coordinate - system being kept fixed and the value of the tensor at one point is compared with its value at another point. Such changes may be usefully regarded as due to the "motion" of the tensor from one point to the other and it is therefore necessary to define some kind of path through the Riemannian space along which the tensor may be "imagined" to travel. These fundamental paths are called **geodesics** of the space and they have

properties analogous to those of straight lines in Euclidean space. The geodesics are particular kinds of curves in the space defined by n equations

$$x^{\lambda} = F^{\lambda}(u), \quad (\lambda = 1,2,3,...,n) \qquad (3.1),$$

where u is parameter varying from point to point of the curve. Substituting equations (3.1) into (1.3.6) $ds^2 = g_{\lambda\nu}\, dx^{\lambda} dx^{\nu}$ and then integrating with respect to u, it is possible to express the *interval* s measured along the curve in terms of u, that is,

$$S = \int_0^1 \{g_{\lambda\nu}(x)(dx^{\lambda}/du)(dx^{\nu}/du)\}^{1/2}\, du \qquad (3.2),$$

where u_o and u_1 are the values of μ at the points P_o and P_1, respectively. The geodesic joining the two points P_o and P_1 is then **defined** to be a curve for which the interval s between P_o and P_1 has a *stationary value* [4]*("stationary interval")* that is, a 4-dimensional "length". *Stationary* here is used in the sense in which that term is used in the calculus of variations, namely, that the *interval* s measured varies minimally along any other neighboring curve joining the two points. Any other curve joining P_o and P_1 , and always lying close to the geodesics, will have equations of the form

$$x^{*\lambda} = x^{\lambda} + \varepsilon\omega^{\lambda} = F^{\lambda}(u) + \varepsilon\omega^{\lambda}(u) \qquad (3.3),$$

where $\omega^{\lambda} = 0$ at $u = u_o$ and $u = u_1$, and ε is a small quantity whose square and higher powers may be neglected. The new interval s* calculated along the neighboring curve joining P_o and P_1 would be given by

$$S^{*} = \int_0^1 \{g_{\lambda\nu}(x^{*})(dx^{*\lambda}/du)(dx^{*\nu}/du)\}^{1/2}\, du \qquad (3.4).$$

Calculating the difference s*- s neglecting all powers of ε above the first order one can verify that[3] to this interval to have a *stationary value* for the geodesic compared with the neighboring curves, s*- s must be zero for any choice of the functions ω^{λ}. In this condition one can prove[3] that the following condition must be satisfied:

$$d^2x^{\tau}/ds^2 + \Gamma_{\lambda\nu}{}^{\tau}\,(dx^{\lambda}/ds)\,(dx^{\nu}/ds) = 0 \quad (\tau = 1,2,3,...,n) \qquad (3.5),$$

where $\Gamma_{\lambda\nu}^{\tau} \equiv (\lambda\mu,\nu) = (\mu\lambda,\nu) = (1/2)\{ \partial g_{\lambda\nu}/\partial x^{\mu} + \partial g_{\nu\mu}/\partial x^{\lambda} - \partial g_{\lambda\mu}/\partial x^{\nu} \}$ are the Christoffel symbols[3,5]. Equations (3.5) are the standard forms of the **geodesics equations** of the Riemannian space, excluding, however, those geodesics for which s = 0 along the curve. As the manifold has n dimension, the geodesic equations are a system of ordinary differential equations for the coordinate variables. If dx^{μ} correspond to an infinitesimal displacement along the geodesic for a change ds of interval, the vector $n^{\mu} = dx^{\mu}/ds$ is called the unit tangent vector to the geodesic.

By dividing (1.3.6) by ds^2 it follows that

$$g_{\mu\nu}\,(dx^{\mu}/ds)(dx^{\nu}/ds) = 1 \qquad (3.6),$$

which shows that n^{μ} is a *unit vector* (vector of unit length). Another important conclusion[3] is that (3.6) is an *integral* of the n equations of the geodesic (3.5)(see also Appendix B**)**.

According to Appendix C a vector q^{λ} is timelike, null or spacelike:

$$\begin{aligned} &\textit{timelike} \text{ when } g_{\lambda\nu}(x)\, q^{\lambda} q^{\nu} > 0, \\ &\textit{null} \text{ when } g_{\lambda\nu}(x)\, q^{\lambda} q^{\nu} = 0, \\ &\textit{spacelike} \text{ when } g_{\lambda\nu}(x)\, q^{\lambda} q^{\nu} < 0. \end{aligned} \qquad (3.7)$$

(3.1)Timelike geodesics.

The equation of motion of a *particle with mass* in GR is written as $Dp^{\mu}/ds = f^{\mu}$, where D is the *covariant derivative* (see Appendix C), $p^{\mu} = mv^{\mu}$, $v^{\mu} = dx^{\mu}/ds$, $ds^2 = g_{\mu\nu}\, dx^{\mu}dx^{\nu}$ and f^{μ} is the *force* acting on the particle, that is,

$$d^2x^{\mu}/ds^2 + \Gamma_{\lambda\nu}^{\mu}\,(dx^{\lambda}/ds)\,(dx^{\nu}/ds) = f^{\mu} \qquad (3.1.1).$$

In the case of free particle for which $f^{\mu} = 0$, Eq.(3.1.1) reduces to

$$d^2x^{\mu}/ds^2 + \Gamma_{\lambda\nu}^{\mu}\,(dx^{\lambda}/ds)\,(dx^{\nu}/ds) = 0 \qquad (3.1.2),$$

which reinforces that the path of a *free massive particle* is a geodesic according to Eq.(3.5) in the spacetime. This result is often stated as a explicit postulate of GR, known as *geodesic postulate*, but emerges here as a natural consequence of the way in which we generalize SR concepts.[8] The path of a free massive particle in flat spacetime of the SR is a straight

line and this generalizes to a geodesic in curved spacetime of the GR.

If, besides (3.1.2), is obeyed the condition $g_{\mu\nu}(dx^\mu/ds)(dx^\nu/ds) = 1$ we say that the geodesics is **timelike.** At any point on the path of a massive particle its world velocity is a tangent vector (n^μ) to the path; Eq.(1.3.6) tell us that this tangent vector is timelike. So, this particle follows a **timelike path** through the spacetime, and in particular a free particle follows a **timelike geodesic.**

(3.2) Null Geodesics.

A **null geodesic**, is obtained by assuming that the interval ds = 0 between any two points on the curve. For instance, from (3.4) we obtain

$$g_{\lambda\nu}(x)(dx^\lambda/du)(dx^\nu/du) = 0 \qquad (3.2.1).$$

This equation in SR generalizes the relation $\eta_{\lambda\nu}(x)(dx^\lambda/dt)(dx^\nu/dt) = 0$, where $\eta_{\lambda\nu} = \text{diag}[-1,1,1,1]$ and the velocity $v^\lambda = (dx^\lambda/dt)$. This is equivalent to $c^2dt^2 - dx^2 - dy^2 - dz^2 = 0$. Trajectories of **light signals** in 3-dim Euclidean space are described by **null geodesics** that are straight lines. In this space $ds^2 = dX^2 + dY^2 + dZ^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = g_{ij}\, dx^i dx^j$, where the "*metric tensor*" $g^{ij} = \delta_{ij}$. As g^{ij} are constants from (3.5) we have $d^2x^\tau/ds^2 = 0$, that is, $d^2X/ds^2 = d^2Y/ds^2 = d^2Z/ds^2 = 0$ which are immediately integrable to give the equations of the **straight line,** passing by the point (X_o, Y_o, Z_o):

$$(X-X_o)/\ell = (Y-Y_o)/m = (Z-Z_o)/n = s, \qquad (3.2.2)$$

where the direction-cosines $dx^\mu/ds = \cos\theta_\mu$ (the unit tangent vectors) of the line are ($\cos\theta_x = \ell$, $\cos\theta_y = m$, $\cos\theta_z = n$) which satisfy the condition, following (3.6),

$$\ell^2 + m^2 + n^2 = 1. \qquad (3.2.3).$$

In Riemannian space **null geodesics,** instead of (3.5), are given by

$$g_{\lambda\nu}(x)(dx^\lambda/du)(dx^\nu/du) = 0 \qquad (3.2.4)$$

and

$$d^2x^\tau/du^2 + \Gamma_{\lambda\nu}{}^\tau\, (dx^\lambda/du)\, (dx^\nu/du) = 0 \quad (\tau = 1,2,3,...,n) \qquad (3.2.5).$$

For a photon that follows a null geodesic Eq.(3.2.4) tell us that the tangent vectors to its path are null. In this case it is more adequate to write (3.2.5) using the fact that the direction of the light propagation in geometric optics is given by the wave quadri-vector $k^\tau = dx^\tau/du$ tangent to the light ray. Thus, (3.2.5) becomes

$$dk^\tau/du + \Gamma_{\lambda\nu}{}^\tau\, k^\lambda\, k^\nu = 0 \quad (\tau = 1,2,3,...,n) \qquad (3.2.6).$$

(3.3)Spacelike Geodesics.

Spacelike paths and **spacelike geodesics** may also be defined following the procedure adopted above, but these have no physical significance unless one believes in tachyons.[8]

Another Examples of Geodesics

(1) Null geodesics in a 3-dim Minkowski space (X,Y,Z) with the metric $ds^2 = dX^2 + dY^2 - dZ^2$

In this case as $g_{11} = g_{22} = 1$ and $g_{33} = -1$, following the procedure adopted in Section (3.2) we obtain

$$\ell^2 + m^2 - n^2 = 0, \text{ instead of (3.2.3).}$$

(2) Great Circles on the Surface of a Sphere.

In a sphere ds^2 is given by (2.1.4): $ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$. Curves with φ = constant are the *great circles* ("**meridians**"); curves with θ = constant are the **small circles** of latitude; only the $\theta = \pi/2$ is a great circle. If in the geodesic equations $d^2x^\tau/ds^2 + \Gamma_{\lambda\nu}{}^\tau (d^2x^\lambda/ds)(d^2x^\nu/ds) = 0$ we put $x^1 = \theta$, $x^2 = \varphi$ and $r = a$ = constant these equations reduce to the pair

$$d[a^2(d\theta/ds)]/ds - a^2 \sin\theta\cos\theta(d\varphi/d\theta)^2 = 0 \quad \text{and} \quad d[a^2\sin^2\theta(d\varphi/ds)]/ds = 0.$$

Particular solutions of these equations are obviously:

(a) φ = constant and $a\theta = s$, corresponding to the "**meridians**".

(b) $\theta = \pi/2$ and $a\varphi = s$, corresponding to the "**equato**r".

Note that circle of latitude on which θ = constant $\neq \pi/2$ does not satisfy the geodesic equations**.** This implies that **great circles** are geodesics and **small circles** are not geodesics.

(3.4)Lagrangian formalism and Geodesics.

Defining the Lagragian function[8] $L = L(\dot{x}^\alpha, x^\alpha)$, with $x^\alpha = x^\alpha(u)$,

$$L(\dot{x}^\alpha, x^\alpha) \equiv (1/2) g_{\mu\nu}(x^\alpha) (dx^\mu/ds) (dx^\nu/ds) \qquad (3.4.1),$$

where $\dot{x}^\alpha = (dx^\alpha/ds)$, the Euler-Lagrange equations are given by

$$d(\partial L/\partial \dot{x}^\mu)/du - \partial L/\partial \dot{x}^\mu = 0 \qquad (3.4.2).$$

From (3.4.1) and (3.4.2) one can deduce the geodesic equations (3.1.2),

$$d^2x^\mu/ds^2 + \Gamma_{\lambda\nu}{}^\mu (dx^\lambda/ds)(dx^\nu/ds) = 0 .$$

This implies the geodesics, timelike or null, can be directly obtained using (3.4.1) and (3.4.2). In Appendix D , using the Lagrangian formalism, we see how to calculate trajectories of particles and photons in the vicinity of spherical massive objects.

(4) Local Cartesian or Local Inertial Coordinate-System.

In a Riemannian space of n dimensions there are certain types of coordinate-systems (X) which are useful for describing the points lying in the neighborhood of a given point O.[3] One example is the *Local Cartesian Coordinate-System* (LCCS) or *Local Inertial Coordinate-System* (LICS) which exists at any point of a Riemannian space of any type. We consider here only the case when the metric is orthogonal, that is,

$$ds^2 = \varepsilon_\lambda \gamma_{\lambda\lambda}(x)(dx^\lambda)^2 \qquad (4.1),$$

where $g_{\lambda\lambda} = \varepsilon_\lambda \gamma_{\lambda\lambda}(x)$ and ε_λ are equal to either +1 or -1. Let the point O have coordinates x_o and consider the coordinates

$$X^\lambda = [\gamma_{\lambda\lambda}(x_o)]^{1/2}(x^\lambda - x_o^\lambda) \quad (\lambda = 1,2,3,...,n) \qquad (4.2),$$

where the summation convention being suspended. The differentials of these coordinates are $dX^\lambda = [\gamma_{\lambda\lambda}(x_o)]^{1/2}dx^\lambda$ and therefore the substitution into (4.1) gives the metric

$$ds^2 = \sum_\lambda \varepsilon_\lambda (dX^\lambda)^2 \qquad (4.3),$$

which is identical with the metric of a *flat space*.[3] The coordinates X^λ are called *local Cartesian* and are valid for those points near enough to O for the differences $[\gamma_{\lambda\lambda}(x)]^{1/2} - [\gamma_{\lambda\lambda}(x_o)]^{1/2}$ to be of order not exceeding $(x^\lambda - x_o^\lambda)$. If the space were indeed flat, then it would be possible to integrate the differential equations $dX^\lambda/dx^\lambda = [\gamma_{\lambda\lambda}(x)]^{1/2}$, a process that cannot be performed in a curved Riemannian space.

As is shown in reference[1] in LICS the Christoffel symbols $\Gamma_{\lambda\nu}{}^\tau$ can be annulled, that is, $\Gamma_{\lambda\nu}{}^\tau = 0$ and, consequently, the geodesic equations (3.5) become $d^2x^\tau/du^2 = 0$ ($\tau = 1,2,3,...,n$). The LIC are also called *locally geodesic coordinates.*

(4.1)Local Inertial Coordinates in Schwarzschild Space-Time.

The metric in the polar Schwarzschild Spacetime (SST) is given by,[1-3,5] where $\chi = GM/c^2$**,**

$$ds^2 = (1-2\chi/r)c^2dt^2 - dr^2/(1-2\chi/r) - r^2d\theta^2 - r^2\sin^2\theta d^2\varphi \qquad (4.1.1).$$

Putting $r = (1 + \chi/2r^*)r^*$, $ct = x^4$, $x^1 = r^*\sin\theta$, $x^2 = r^*\sin\theta\sin\varphi$, $x^3 = r^*\cos\theta$ the metric ds^2 given by (4.2), at the event $(x_o^4, x_o^1, x_o^2, x_o^3)$ introducing the LIC defined by

$$X^4 = [(1-\chi/2r_o^*)/(1+\chi/2r_o^*)](x^4 - x_o^4) \quad \text{and} \quad X^i = (1+\chi/2r_o^*)^2(x^i - x_o^i),$$

becomes[3]

$$ds^2 = (dX^4)^2 - \sum_i (dX^i)^2 \qquad (4.1.2).$$

In the case of the Minkowski Space -Time (MST) where the metric is given by $ds^2 = (dx^4)^2 - \sum_i (dx^i)^2$ the velocity q_i of the particle is defined by $q_i = c[\sum_i (dx^i)^2]^{1/2}/dx^4$.[3] Hence at the event (x_o), the *local Cartesian velocity* q_o of the particle may be defined in manner analogous by

$$q_o = \{c[\sum_i (dX^i)^2]^{1/2}/dX^4\}_o = [(1+\chi/2r_o^*)^3/(1-\chi/2r_o^*)]\{\sum_i c(dx^i)^2/dx^4\}_o \quad (4.1.3).$$

LIC may be set up at any event and one may also work back to the polar system (t,r,θ,φ).

(5) Earth Gravitational Field (Schwartzschild Metric).

Let us assume that the Earth is a spherical body with mass M, radius R and with angular velocity Ω. Taking the Earth center of mass as the origin of an inertial system Σ one can show that the gravitational field generated by the Earth in vacuum (disregarding the Earth's spin effect) is described by the Schwartzschild metric(SM)[1-3,5] ds^2 that in spherical polar coordinates (r, θ, φ) is given by

$$ds^2 = (1-2\chi /r)\, c^2dt^2 - dr^2/ (1-2\chi/r) - r^2d\theta^2 - r^2\sin^2\theta d^2\varphi \qquad (5.1),$$

where $\chi = GM/c^2$. It is always emphasized that in GR the variables t and r are coordinates, that is, they are not necessarily "physical quantities" such as *time* and *distance* in classical mechanics. In Appendix E we analyze the Length and Time in the SST.

(5.1) Gravitational Doppler Effect.

Consider two clocks 1 and 2 at rest with respect to Earth with coordinates r_1 and r_2. As for both $dr = d\varphi = d\theta = 0$ using (5.1) we obtain $ds^2 = c^2d\tau^2$ where we made $t \equiv \tau$, defined as proper time [1-3,5] that is, the time measured in the reference frame in which the clock is at rest. Thus, for points 1 and 2,we get

$$d\tau_1 = \sqrt{g_{oo}}\,(r_1)\, dt = (1-2Gm/c^2r_1)^{1/2}\, dt$$

$$d\tau_2 = \sqrt{g_{oo}}\,(r_2)\, dt = (1-2Gm/c^2r_2)^{1/2}\, dt \qquad (5.1.1)$$

When $r \rightarrow \infty$ the proper time τ becomes equal to the time coordinate t. Using (5.1) we can compare the time intervals that would be measured in reference frames 1 and 2. According to (5.1.1), if $r_2 > r_1$ the clock in 2 moves faster than the clock in 2. Time measurements made with atomic

clocks of high precision have confirmed the predictions made by (5.1.1).[7] In these clocks the time measurements are based on the frequencies of vibrations ν of atomic transitions. Thus, in 1 and 2 we would have, respectively, the frequencies $\nu_1=1/d\tau_1$ and $\nu_2=1/d\tau_2$ given by, where $\nu = 1/dt$,

$$\nu_1 = \nu/(1-2Gm/c^2r_1)^{1/2} \quad \text{and} \quad \nu_2 = \nu/(1-2Gm/c^2r_2)^{1/2}$$

that is,

$$\nu 1/\nu 2 = \{(1-2Gm/c^2r_2) / (1-2Gm/c^2r_1)\}^{1/2} \qquad (5.1.2).$$

Equations (5.1.2) show the effect of the change of the electromagnetic frequencies generated by the gravitational field. It is the *Gravitational Doppler Effect.*

(5.2) Gravitational & Transverse Kinematic Doppler Effect.

To evaluate the change in frequency ν due to the gravitational field and to the speed of movement of the clocks let us assume that one clock is fixed on Earth and the other is rotating around the Earth. Take clock 1 fixed on the surface of the Earth. Thus, 1 has a rotational motion with angular velocity Ω relatively to the inertial system Σ. Let us assume that 2 is fixed on an airplane that carries out a circumnavigation movement around the Earth (for example, along the terrestrial equator) with a constant tangent velocity V. The fixed clock 1 on Earth (reference 1) putting $\Omega = d\theta/dt$ and $dr/dt = 0$, using (5.1) would measure a time interval $d\tau^1 = d\tau^T$ given by

$$(d\tau^T)^2 = \{(1-2GM/c^2R) - R^2\Omega^2/c^2\}dt^2 \qquad (5.2.1).$$

It is important to note that in (5.2.1) besides the *gravitational effect* given by the term $(1-2GM/c^2R)^2dt^2$ there is also a kinematic effect given by $-(R^2\Omega^2/c^2)\, dt^2$ which in SR is responsible for the *Transverse Doppler Effect*. For the fixed reference 2 in airplane we have $r_2 = R + h$, where h is the height of the flight; as the airplane moves with velocity V relative to the ground its velocity U relative to the inertial frame Σ is given by the Lorentz Transformation: [1,2,5]

$$U = [(R + h)\,\Omega + V] / [1+ \Omega\,(R + h)V/c^2] \qquad (5.2.2).$$

As in these flights $c^2 >> R\Omega V$ [2] the we have $U \approx (R + h)\Omega + V$; so the time interval $d\tau^2 = d\tau_A$, measured by a fixed clock in the plane is given by

$$(d\tau_A)^2 = \{[1-2GM/c^2(R + h)] - [(R + h)\Omega + V]^2/c^2\}\, dt^2 \qquad (5.2.3),$$

where the velocity $V \approx 300$ m/s has signals ± depending on the direction of movement of the airplane in relation to the ground. Measurements using high precision clocks have confirmed[2,7] the predictions given by (5.2.1) and (5.2.3). We recall that in SR the *transverse Doppler effect* was introduced "ad hoc" to explain the kinematic effect because the reference system where clock 1 is fixed was non-inertial with an radial acceleration $a_R = \Omega^2 R$. From this we verified that this kinematic effect can only be explained rigorously within the context of the GR.

(5.3) Gravitational & Longitudinal Kinematic Doppler Effect.

Now let's consider that a light detector (clock) is fixed at a point P_o and that a light emitter at a point P is moving directly away or approaching P_o (see Figure 1). The **radial Cartesian velocity** of P defined by $q = ds/dt$, according to (5.1), would be given by[3]

$$q = \{(dr/dt)^2/(1-2\chi/r) + r^2(d\theta/dt)^2 + r^2\sin^2\theta(d\varphi/dt)^2\}^{1/2}/(1-2\chi/r)^{1/2} =$$

$$= (1-2\chi/r)^{-1/2} \{(1-2\chi/r)^{-1}(dr/dt)^2 + r^2(d\theta/dt)^2 + r^2\sin^2\theta(d\varphi/dt)^2\}^{1/2} \quad (5.3.1)$$

Defining the *radial coordinate velocity* by $V = dr/dt$ and assuming that $d\theta = d\varphi = 0$ we get from (5.3.1):

$$q = (1-2\chi/r)^{-1/2}\, V \qquad (5.3.2),$$

showing that $q \rightarrow V$ when $r \rightarrow \infty$ or $M \rightarrow 0$ in absence of matter. It can also be obtained[3] the *general γ relation*,

$$\gamma^2 = (1 - 2\chi/r)/(1 - q^2/c^2) \qquad (5.3.3).$$

We see that $\gamma = (1-V^2/c^2)^{1/2} = (1-\beta^2)^{1/2}$, with $\beta = V/c$, when $r \rightarrow \infty$ or $M \rightarrow 0$.

Let us suppose that the from the point P are emitted two pulses of

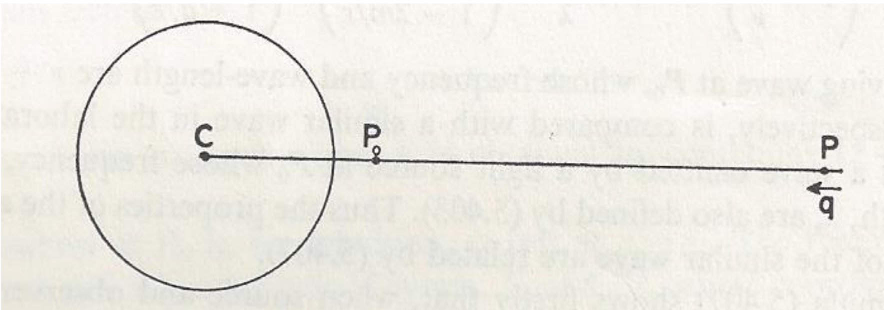

Figure 1. Light detector fixed at P_o and light emitter in motion at P.

light with own frequency ν at the events E and E´ defined in the coordinate system Σ by E ≡ (t, r, θ, φ) and E´≡ (t + dt, r + dr, θ, φ). They are separated by the invariant proper-time interval

$$ds^2 = (1-2\chi/r)c^2dt^2 + dr^2/(1-2\chi/r)$$
$$= (1-2\chi/r)(1 - q^2/c^2)dt^2 \qquad (5.3.4),$$

where the radial velocity q is given by $q = (dr/dt)/(1-2\chi/r)$, according (4.1.4) putting dφ/dt = dθ/dt = 0. The reception of the first signal by (fixed) P_o is defined by the event E_o defined by $E_o \equiv (t_o, r_o,0,0)$. The reception of the second signal is defined by $E_o' \equiv (t_o + dt_o, r_o, 0, 0)$. These two events are separated by the interval ds_o given by

$$ds_o = (1-2\chi/r_o)^{1/2}\, cdt_o \qquad (5.3.5),$$

and therefore the frequency, $\nu_o + d\nu_o$, and the wavelength, $\lambda_o + d\lambda_o$, received by detector P_o are given by

$$\nu_o + d\nu_o = c/ds_o \quad \text{and} \quad \lambda_o + d\lambda_o = cds_o \qquad (5.3.6).$$

Since the events E & E_o and E' & E_o' are connected by light signals they can be taken as *null geodesics* through P and P_o putting ds = dθ = dφ = 0 in (5.3.1), obtaining $(1-2\chi/r)\, c^2dt^2 + dr^2/(1-\chi/r) = 0$, that is,

$$dr/dt = c(1-2\chi/r) \qquad (5.3.7).$$

So, for the pair E&Eo we have, integrating (5.3.7) with $t_o \rightarrow t$ and $r_o \rightarrow r$:

$$t - t_0 = (1/c)\int_{r_0}^{r} dr/(1-2\chi/r)$$

(5.3.8).

Similarly, for the pair E´& E_o´ integrating (5.3.4) with $t_o+dt_o \rightarrow t+dt$ and $r_o+dr_o \rightarrow r+dr$ we get

$$t + dt - t_0 + dt_0 = (1/c)\int_{r_0+dr_0}^{r_0} dr/(1-2\chi/r)$$

(5.3.9).

Subtracting (5.3.8) and (5.3.9) results $dt_o - dt = -(1/c)\, dr/(1-2\chi/r) = -(q/c)dt$, gives

$$dt_o = (1- q/c)dt \qquad (5.3.10).$$

In this way, from (5.3.5) , (5.3.9) e (5.10) we obtain,

$$ds_o/ds = \{(1- 2\chi/r_o)/(1- 2\chi/r)\}^{1/2} \{(1 - q/c)/(1 + q/c)\}^{1/2} \qquad (5.3.11)$$

Since $\nu = 1/ds$ (5.3.11) can also be written as

$$\nu/\nu_o = \{(1- 2\chi/r_o)/(1- 2\chi/r)\}^{1/2} \{(1 - q/c)/(1 + q/c)\}^{1/2} \qquad (5.3.12).$$

Note that ν_o is the frequency measured by a detector (P_o) fixed at the Earth and ν is the *proper frequency* emitted by a system (P) in motion with the radial velocity q relatively to the Earth (P_o).

From (5.3.12) we see that the gravitational and kinematic Doppler Effects are intertwined. (5.3.12) predicts only kinematic effects by making $G = \chi = 0$ and only gravitational effects when $q = 0$.

Emitter and receiver fixed at points E and R.[8]

In this particular case the two points $E \equiv (t_E,r_E,\theta_E,\varphi_E)$ and $R \equiv (t_R,r_R,\theta_R,\varphi_R)$ are connected by a *null geodesics* defined by the equation

$$(1-2\chi/r)c^2 t^{+2} - (1-2\chi/r)^{-1} r^{+2} - r^2\theta^{+2} - r^2\sin^2\theta\, \varphi^{+2} = 0 \qquad (5.3.13),$$

where, generically, $f^+ = df/du$. From (5.3.13) we have obtain

$$dt/du = (1/c)\{(1-2\chi/r)^{-1} g_{ij} (dx^i/du)(dx^j/du)\}^{1/2}, \qquad \text{where } g_{ij} = - g_{ij}.$$

Integrating this expression we get

$$t_R - t_E = (1/c) \int\{(1-2\chi/r)^{-1} g_{ij} (dx^i/du)(dx^j/du)\}^{1/2} du.$$

As this integral depends only on the path through space, so with a spatially fixed emitter and spatially fixed receiver, $t_R - t_E$ is the same for all signals, 1and 2, sent. Thus, for two signals we have $t_R^{(1)} - t_E^{(1)} = t_R^{(2)} - t_E^{(2)}$ showing that time intervals measured in E are equal to that measured in R, that is,

$$\Delta t_R = t_R^{(2)} - t_r^{(1)} = t_E^{(2)} - t_E^{(1)} = \Delta t_E \qquad (5.3.14).$$

That is, the coordinate time difference at the point of emission E equals the coordinate time difference at the point R of reception. However, the clocks

of an observer at R and of an emitter at E record *proper times* and not *coordinate times*. The record proper times $\Delta\tau_R$ and $\Delta\tau_E$ are given by [8]

$$\Delta\tau_R = (1-2\chi /r_R)^{1/2}\,\Delta t_R \quad \text{and} \quad \Delta\tau_E = (1-2\chi /r_E)^{1/2}\,\Delta t_E \qquad (5.3.15).$$

Since $\Delta t_R = \Delta t_E$ we have

$$\Delta\tau_R /\Delta\tau_E = [(1-2\chi /r_R)/(1-2\chi /r_E)]^{1/2} \qquad (5.3.16),$$

that is a particular case of (5.3.12) when q = 0, remembering that $\Delta\nu = 1/\Delta\tau$.

(5.4) Particle Geodesics in Weak Gravitational Field Limit.

Let us see recover the Newtonian equations of motion of a massive particle using as a starting point the geodesic equations in the limit of very weak gravitational field (Φ) and with velocities $dx^i/dt \ll c$ (i =1,2,3). Suppose a system where the metric tensor is given by $g_{\lambda\nu} = \eta_{\lambda\nu} + h_{\lambda\nu}$ where $\eta_{\lambda\nu}$ = diag[-1,1,1,1] and $h_{\lambda\nu}$ is very small but cannot be neglected. In these way one can show, after a somewhat long calculation, that[8]

$$md^2x^i/dt^2 = - m\delta^{ij}\partial_j(c^2h_{oo}/2) + mc\delta^{ik}(\partial_j h_{ok} - \partial_k h_{oj})(dx^j/dt) \qquad (5.4.1).$$

The first term on the right would be the gravitational force -m grad(Φ) where $\Phi = c^2h_{oo}/2$ + const and the second term would be a Coriolis force that is null in a nonrotating reference system. Taking const = 1 we see that g_{oo} in the Newtonian approximation would be given by $g_{oo} = 1 + 2\Phi/c^2$.

Example 1.Newton´s Law of Universal Gravitation.

Using (5.4.1) we recover Newton´s law remembering that g_{oo} in the SM in the weak field limit (for large $1/\chi$) in given by $g_{oo} = (1-2\chi /r) \approx 1 - 2GM/rc^2$, that is, $h_{oo} = - 2GM/rc^2$. This would imply that V = - GM/r and consequently, using (5.4.1):

$$\mathbf{F}(r) = md^2\mathbf{r}/dt^2 = -m\,\mathrm{grad}(V) = - (GMm/r^2)\,\boldsymbol{r}$$

in agreement with the gravitational Newton´s law.

(6) Principle of Equivalence and Geodesic Principle.

In many basic textbooks[1-3,5] about the Theory of Relativity are found analysis on the Principle of Equivalence. This is proposed as an immediate consequence of recent experimental measurements[2] which

show that the gravitational mass m_g is equal to the inertial mass m_i with an accuracy of one part in 10^9. We suggest the lecture, for instance, of the analysis presented by Yilmaz [5] where is shown Figure 2:

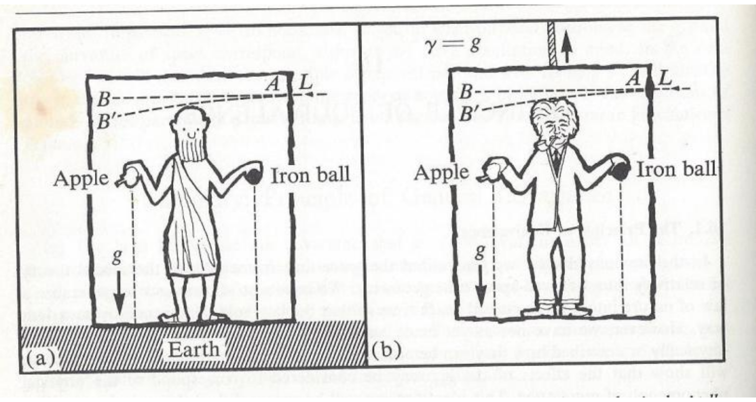

Figure 2. A system **(a)** which is stationary in a uniform gravitational field g is physically equivalent to a system which is in a gravitational-free space **(b)** but accelerated in the opposite direction with an acceleration $\gamma = g$.

The Principle of Equivalence can be written, for instance, as:

(6.a) "It is impossible to tell whether a system (room) is in a state of accelerated motion by experiments performed in that system alone. For the observer in the room can claim that this room is stationary but there is a gravitational field present in the room."

(6.b)"A system which is stationary in a gravitational field of strength *g* is physically equivalent to a system which is in a gravitation-free space but accelerated in the opposite direction with an acceleration *g*."

(6.c) " In physical terms the effects of a gravitational field can be removed, locally at least, by employing an appropriately chosen *accelerated coordinate system*. This is a local coordinate system which is falling freely in the gravitational field of the distribution of matter".

According to **Section 4** it is always possible to determine a Locally Inertial coordinate-system (LIC) at any point of a Riemannian space. The metric at the origin of this coordinate-system is of a *flat space* (Appendix A). This implies that the *Riemann-Christoffel* $R^{\sigma}{}_{\lambda\nu\mu}$, the *Ricci* $R_{\lambda\mu}$ and the Curvature Invariant $R^{\lambda\lambda}$ tensors are equal to zero (see Appendix A). In any flat Riemannian space-time (x), representing a large distribution of matter, the motion of a small particle will be described by the *time-like* geodesics

$$d^2x^\sigma/ds^2 = 0 \qquad (\sigma = 1,2,3,4) \qquad (6.1),$$

which represent the motion of a particle locally identical with the motion under Newton´s Fist law. They are therefore immediately integrable to give $dx^\sigma/ds = v^\sigma$ and $x^\sigma - x_o^\sigma = s\, v^\sigma$ where the four v^σ and the four x_o^σ are constants of integration. In physical terms, these LIC may be said to form a system which is falling freely in the gravitational field of the distribution of matter which corresponds to the formulation **(6.c)** of the Principle of Equivalence. In other words, the existence of a LICS at every event in space-time expresses the Principle of Equivalence and indicates that the effect of a gravitational field can be abolished locally by the choice of a suitable accelerated coordinate-system. The representation of the history of the motions of *small* test-particles by time-like geodesics and of light-rays by null-geodesics (see **Section 3.3**) in Riemannian spaces form the "**Geodesic Principle".** As stated above, the geodesic principle presupposes that the distribution of matter is given through its energy-tensor, that the appropriate Riemannian space-time has been determined through Einstein´s equations and that the particle itself contributes nothing to the distribution under whose gravitational influence it moves.[3] See comments of McVittie[3] about geodesic principle, continuous distributions of matter, cosmology and Einstein´s equations.

(7)Accelerated Coordinate-Systems in Special Relativity.

It has been mentioned above that the existence of a LICS at every event in space-time expresses the Principle of Equivalence and indicates that the effect of a gravitational field can be abolished locally by the choice of a suitable accelerated coordinate-system. This result has led to some confusion for it has been interpreted as equivalent to the statement that the essence of GR consists in the use of accelerated coordinate-systems, in contrast with SR, where relatively non-accelerated coordinate-systems are employed.[3] This statement is, however, misleading because there is nothing to prevent the investigator from using accelerated coordinate-systems is SR if he chooses to do so. Let us consider, for instance, an inertial system **S** in the Minkowski space-time (cT,X,Y,Z), with metric

$$ds^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2 \qquad (7.1).$$

In **S** the motion of a non accelerated the test particle would be governed by the time-like geodesics equations $d^2X^\mu/ds^2 + \Gamma_{\lambda\nu}{}^\mu (dX^\lambda/ds)(dX^\nu/ds) = 0$, where the Christoffel symbols $\Gamma_{\lambda\nu}{}^\mu = 0$, that is,

$$d^2X^\mu/ds^2 = 0 \quad (\mu = 1,2,3,4). \tag{7.2}$$

The Riemann-Christoffel and the Ricci tensors in S are identically zero and so is the energy-tensor.

Consider now a non-inertial Minkowski space-time coordinate-system **S´** in which the (ct,x,y,z) are obtained by Lorentz transformations from (cT,X,Y,Z). These transformations do not alter the null-character of the Riemann-Christoffel and the Ricci tensors of the MST which remains a flat space-time (see Appendix A). Hence the Ricci tensor is still identically zero and so is the energy-tensor, in spite of the fact that the coefficients $g_{ik}(x_i)$ of the new metric $-ds^2 = g_{ik}(x_i)\, dx_i\, dx_k$ can be no longer constants. Thus, ds^2 **does not represent** any distribution of matter and therefore no gravitation field is present. Note that the equations of motion of a particle in S´ can be difficult to obtain in, general case, when the Lorentz $\gamma \neq 1$. An example of this can be seen, for instance, in reference [3], pag.78. In Sections (7.1) and (7.2) are shown simple cases when $\gamma = 1$.

(7.1) Rotating Reference System.

(7.1.a) **Cartesian Coordinates.**

Let us pass from an inertial reference frame S ≡ (T, X, Y, Z) to a non-inertial rotating referential system S´ with constant angular velocity ω along the z-axis.

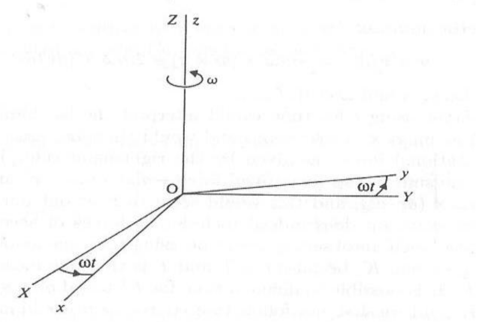

Figure 2. Coordinate system S´ ≡ (ct, x, y, z) rotating relative to the coordinate system S ≡ (cT, X, Y, Z).

Defining a new referential system S´ ≡ (t, x, y, z):

$T = t$,
$X = x \cos(\omega t) - y \sin(\omega t)$,
$Y = x \sin(\omega t) + y \cos(\omega t)$,
$Z = z$,

we see that $ds^2 = c^2dT^2 - dX^2 - dY^2 - dZ^2$ now becomes written as

$$ds^2 = c^2d\tau^2 = [c^2 - \omega^2(x^2+y^2)]dt^2 + 2\omega x\, dy\, dt - dx^2 - dy^2 - dz^2 \quad (7.1.a1).$$

For S´ the geodesic equations are $d^2x^\tau/d\tau^2 + \Gamma_{\lambda\nu}{}^\tau (dx^\lambda/d\tau)(dx^\nu/d\tau) = 0$, where $\Gamma_{\lambda\nu}{}^\tau = (1/2)\{ \partial g_{\lambda\nu}/\partial x^\mu + \partial g_{\nu\mu}/\partial x^\lambda - \partial g_{\lambda\mu}/\partial x^\nu \}$, $(\tau = 0,1,2,3)$. Since $g_{oo} = [c^2 - \omega^2(x^2+y^2)]$, $g_{o2} = 2\omega x$ and $g_{11} = g_{22} = g_{33} = -1$ we see that the only non null $\Gamma_{\lambda\nu}{}^\tau$ are $\Gamma_{00}{}^1 = -\omega^2 x$, $\Gamma_{20}{}^1 = -2\omega$, $\Gamma_{00}{}^2 = -\omega^2 y$ and $\Gamma_{10}{}^2 = 2\omega$. Thus, the equations of motion for a free massive particle are given by,[8]

$$d^2t/d\tau^2 = 0 \;,$$
$$d^2x/d\tau^2 - \omega^2 x(dt/d\tau)^2 - 2\omega(dy/d\tau)(dt/d\tau) = 0, \qquad (7.1.a2)$$
$$d^2y/d\tau^2 - \omega^2 y(dt/d\tau)^2 + 2\omega(dx/d\tau)(dt/d\tau) = 0,$$
$$d^2z/d\tau^2 = 0.$$

As the first equation implies that $dt/d\tau = k$ = constant, that is, $dt = k\, d\tau$ the remaining (7.1.2) equations may be written as

$$d^2x/dt^2 - \omega^2 x - 2\omega(dy/dt) = 0,$$
$$d^2y/dt^2 - \omega^2 y + 2\omega(dx/dt) = 0, \qquad (7.1.a3),$$
$$d^2z/dt^2 = 0$$

These calculations clearly shown that in the non inertial system S´ the **acceleration has a geometrical nature**. Multiplying by m the **noninertial accelerations** given by Eqs.(7.1.a3), written in 3-dim vector notation, we get the **noninertial forces**

$$m d^2\mathbf{r}/dt^2 = m\, \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - 2m\boldsymbol{\omega} \times (d\mathbf{r}/dt) \qquad (7.1.a4),$$

where $\mathbf{r} \equiv (x,y,z)$ and $\boldsymbol{\omega} \equiv (0,0,\omega)$. The component $2m\boldsymbol{\omega} \times (d\mathbf{r}/dt)$ is the centrifugal force and $- 2m\boldsymbol{\omega} \times (d\mathbf{r}/dt)$ the Coriolis force.[1,9] . These forces **are not due to physical interactions**. Note that with the choice $T \equiv t$, where T is the proper time measured by clocks at rest in S, the time t is **exactly the proper time** for an observer situated at the common origin O of the two systems. So, observers close to O who are at rest in the rotating system would accept Eqs.(7.1.a3) and (7.1.a4) as approximately valid and

recognizes the terms on the right as forces of acceleration.[8]

Similar results can be deduced using the "weak field approximation" seen in **Section (5.4)**, when we have assumed that $g_{\lambda\nu} = \eta_{\lambda\nu} + h_{\lambda\nu}$ where $\eta_{\lambda\nu}$ = diag[-1,1,1,1] and $h_{\lambda\nu}$ is very small but cannot be neglected. The terms $h_{\lambda\nu} \rightarrow - \omega^2(x^2+y^2)/c^2$, $\omega y/c$, $- \omega x/c$ would be small for small angular velocities ω and so, (7.1.a3) and (7.1.a4) can be obtained.[8]

(7.1.b)Cylindrical Coordinates.

Transforming Cartesian (t,x,y,z) to cylindrical coordinates putting $z' = z$, $x = \rho \cos\varphi$, $y = \rho \sin\varphi$ from (7.1.a2) we get the "accelerations"

$$
\begin{aligned}
&d^2t/d\tau^2 = 0 \ , \\
&d^2\rho/d\tau^2 - \omega^2\rho(dt/d\tau)^2 - \rho(d\varphi/dt)^2 - 2\omega\rho(d\varphi/d\tau)(dt/d\tau) = 0, \\
&d^2\varphi/d\tau^2 - \omega^2\rho^{-1}(d\rho/d\tau)(d\varphi/d\tau) + 2\omega\rho^{-1}(d\rho/d\tau)(dt/d\tau) = 0, \qquad (7.1.b1) \\
&d^2z/d\tau^2 = 0.
\end{aligned}
$$

Introducing the mass m of the particle and rearranging we have the virtual forces, noting that $dt = k\, d\tau$,

$$
\begin{aligned}
&(A)\ m[d^2\rho/d\tau^2] = \rho[(d\varphi/dt)^2 + m\omega^2] + 2m\omega\rho(d\varphi/d\tau), \\
&(B)\ m[d^2\varphi/d\tau^2] = - 2m(d\rho/d\tau)(d\varphi/d\tau)/\rho - 2m\omega(d\rho/d\tau)/\rho, \qquad (7.1.b2). \\
&(C)\ m(d^2z/d\tau^2) = 0,
\end{aligned}
$$

The first term of (A) is the *centrifugal force* due to particle´s own motion and of the rotation of the frame and the second term is the *radial Coriolis force*. The right hand side of (B) is the *tangential Coriolis force*; it depends on the radial velocity, and on the angular velocity ω of the particle in the frame and rotation of the frame.

Now, if the inertial cylindrical coordinate system is represented by $S' = (t, \rho', \varphi', z')$ we have

$$ds^2 = c^2dt^2 - d\rho'^2 - \rho'^2 d\varphi'^2 - dz'^2 \qquad (7.1.b3).$$

If t, ρ, φ and z are the cylindrical coordinates of the rotating system S = (t, ρ,φ, z) we see that $\rho' = \rho$, $z' = z$ and $\varphi' = \varphi + \omega t$. In this way (7.1.b3) becomes given by

$$ds^2 = (c^2 - \omega^2\rho^2)dt^2 - 2\omega\rho^2 d\varphi dt - dz^2 - \rho^2 d\varphi^2 - d\rho^2 \qquad (7.1.b4).$$

(8) Distances and Time Intervals in GR.

In GR the choice of a reference system is not limited by nothing; the 3 coordinates x^1, x^2, x^3 can be arbitrary quantities defining the position of the particles in the space and the temporal coordinate x^o can be determined by a clock marking its *proper time*. The main problem is how one can determine using x^o, x^1, x^2, x^3 the *real distances* ℓ and the *real time lapses* τ.

(8.a) Real time intervals.

Let us first determine the connection between *real times* τ and the *coordinates* x^o. To do this let us estimate the interval ds between two events that occur in the same point, that is, when $dx^1 = dx^2 = dx^3 = 0$. So, as $- ds^2 = g_{ik}\, dx_i dx_k$ we get $ds^2 = - c^2\, d\tau^2 = g_{oo}\, dx_o^{\,2}$, that is

$$d\tau = (1/c)(- g_{oo})^{1/2}\, dx^o \qquad (8.a.1),$$

from which we see that the finite *real time interval* τ between two arbitrary events occurring in the same point of the space is given by,

$$\tau = (1/c) \oint (-g_{00})^{1/2}\, dx^0 \qquad (8.a.2).$$

The relations (8.a.1) and (8.a.2) show how to determine the real times τ (that is, the *proper times* in one point of the space) as a function of the coordinate x_o. Note that, to have physical meaning, we must have

$$g_{oo} < 0 \qquad (8.a.3).$$

It is important to stress that the spatial components of tensor g_{ij} must be *positive* and that g_{oo} must be negative. One metric tensor that does not satisfy the last condition (8.a.3) cannot represent a *real* gravitational field. If (8.a.3) is not obeyed would signify that the correspondent reference systems could not describe real bodies.[1]

Weak gravitational field.

As will be seen in **Section (5.4)** in weak gravitational fields Φ(**r**) we have at a given point of the space, putting $x^o/c = t$:

$$\tau = (x^o/c)(- g_{oo})^{1/2} \approx (x^o/c)\, [1+ 2\Phi/c^2\,]^{1/2} \approx (1 + \Phi/c^2)\, t \qquad (8.a.4).$$

When $\Phi < 0$ the *real* or *proper time* τ flows more slowly than the *absolute*, *universal* or *coordinate time* $t = (x^o/c)$ which is measured in absence of gravitational fields.

In the Newtonian approximation, according to Section 5, g_{oo} in would be given by $g_{oo} = (1\text{-}2\chi /r) = 1 + 2\Phi/c^2 \approx 1 - 2GM/rc^2$. If, for one is measuring the time fixed on the Earth surface, taking G ~ 6,7 10^{-11} MKS, M ~ 6 10^{24} kg, r ~ 6,4 10^6 m and c ~3 10^8 m/s using (8.a.4) he will see that $\tau/t \sim 1\text{-}10^{-10}$. That is, the time measured on the earth flows more slowly than the time measured in absence of the gravitational field.

***Time Synchronization*.**

Now let us present the notion of **Simultaneity** in GR. That is, let us see if it is possible to *synchronize* watches that are in two different points of the space, in other words, to obtain a correspondence between their indications. Let us consider the general case when the metric tensor depends on the time coordinate x^o . Such *synchronization* must be done, evidently, by an exchange of luminous signals between these two points. To do this let us consider the process of propagation of light signals between two *infinitelly* close points A and B represented by Figure 3. If x_o is the time of arrival of the signal in A, the moments of its departure from B and its arrival in B will be, respectively, $x_o + dx_o^{(1)}$ and $x_o + dx_o^{(2)}$.

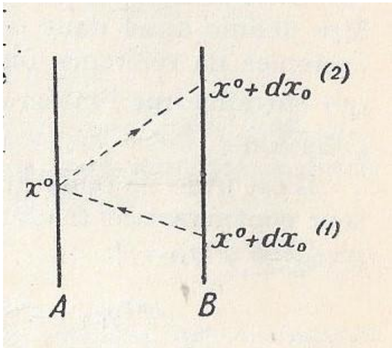

Figure 3. Propagation of ligth signals between two infinitesimally close points A and B.

In Figure 3 the parallel straight lines represent the universe lines corresponding to the coordinates x^α and $x^\alpha + dx^\alpha$, and the dotted sections represent the universe lines of the luminous signals(to simplify we can suppose that $dx_o^{(1)} < 0$ and $dx_o^{(2)} > 0$). Putting $- ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta + 2g_{o\alpha}\, g_{oo}dx^o dx_\alpha + g_{oo}dx_o^2$ we have

$$dx_o^{(1)} = -\{ g_{o\alpha}dx^\alpha - [(g_{o\alpha}\, g_{o\beta} - g_{\alpha\beta}g_{oo})\, dx^\alpha dx^\beta]^{1/2} \}/g_{oo}$$

$$dx_o^{(2)} = -\{ g_{o\alpha}dx^\alpha + [(g_{o\alpha}\, g_{o\beta} - g_{\alpha\beta}g_{oo})\, dx^\alpha dx^\beta]^{1/2} \}/g_{oo} \qquad (8.a.5)$$

It is clear that the total lapse of "time" between the emission and the return of the signal to the same point B is equal to

$$dx_o^{(2)} - dx_o^{(1)} = -(2/g_{oo})\, [(g_{o\alpha}\, g_{o\beta} - g_{\alpha\beta}g_{oo})\, dx^\alpha dx^\beta]^{1/2} \qquad (8.a.6).$$

Let us stipulate as simultaneous with the instant x^o at the point A the indication χ^o of the clock at B located at the half of the instants of emission and return of the signal in this point, that is

$$\chi^o(B) = x^o + \Delta x^o = x^o(A) + (1/2)[dx_o^{(2)} + dx_o^{(1)}] \qquad (8.a.7).$$

In other words, $\chi^o(B)$ is "simultaneous" with $x^o(A)$. Using (8.a.6) we obtain

$$\Delta x^o = - g_{o\alpha}dx^\alpha/g_{oo} \qquad (8.a.8),$$

which is the "time" difference between two simultaneous events happening at points infinitesimally distant A and B. Since $t = cx^o$ (8.a.7) can also be written, according to Einstein, as

$$t(B) = t_1^A + (t_2^A - t_1^A)/2 = (t_2^A + t_1^A)/2 \qquad (8.a.9),$$

where $t_2^A = c(x^o + dx_o^{(2)})$ and $t_1^A = c(x^o - dx_o^{(1)})$.

In *stationary fields* that is, when the metric tensor depends on the coordinate time x^o the clocks *synchronization in all space is impossible*. For *static fields,* that is, when the metric is time independent this is possible. In this case, using (8.a.8), we can determine the simultaneity of two events performing the integration

$$\Delta x^0 = -\oint g_{0\alpha}\, dx^\alpha / g_{00}$$

(8.a.10).

In stationary fields is possible to synchronize clocks only along a closed path, returning to the initial starting point, obtaining the time difference Δt by the cyclic integration[1] of (8.a.10):

$$\Delta t = -(1/c)\oint g_{0\alpha}\, dx^\alpha / g_{00}$$

(8.a.11).

In particular, in the case of a rotating coordinate system, taking into account the metric given by (7.1.b4), we have,

$$\Delta t = (1/c^2)\int \omega\rho^2 d\varphi /(1-\omega^2\rho^2/c^2) \qquad (8.a.12).$$

(8.b) Real Spatial distances.

In SR one can define the element $d\ell$ of real spatial distance as the interval between two events spatially close dx^σ (σ =1,2,3) happening at the same "instant" x^o, that is, putting $dx^o = 0$ in $-ds^2 = g_{\lambda\nu}dx^\lambda dx^\nu$. In GR it cannot in general be done because the proper time in a gravitational field is differently connected with the coordinate x^o in different points of the space, according to (8.a.1) and (8.a.2). The calculation of the element $d\ell$ is given, for instance, by [1,10]

$$d\ell^2 = \gamma_{\alpha\beta}dx^\alpha dx^\beta \quad \text{and} \quad \gamma_{\alpha\beta} = g_{\alpha\beta} - (g_{o\alpha}\, g_{o\beta})/g_{oo} \qquad (8.b.1),$$

where in the product $dx^\alpha dx^\beta$ we have only α and β = 1,2,3. That is, temporal components do not contribute.

Example. Cylindrical Coordinates Rotation.

Using the cylindrical metric shown in Section 7 and (8.b.1) we get

$$ds^2 = -(1 - \chi^2)dx_o^2 + dx_1^2 + (2\rho\chi)dx_2dx_o + x_1^2dx_2^2 + dx_3^2 \qquad (8.b.2),$$

where $ct = x_o$, $\rho = x_1$, $\varphi = x_2$ e $z = x_3$ and $\chi = \omega\rho/c$. Since the product dx_2dx_o are not taken into account we have the non null terms:

$$g_{oo} = -(1 - \chi^2),\ g_{11}= g_{rr} = 1,\ g_{22} = g_{\varphi\varphi} = \rho^2 \text{ and } g_{33} = g_{zz} = 1.$$

In this way, from $\gamma_{\alpha\beta} = g_{\alpha\beta} - (g_{o\alpha}\, g_{o\beta})/g_{oo}$ we get $\gamma_{11} = g_{11}-(g_{o1}\, g_{o1})/g_{oo}= 1$, $\gamma_{22} = g_{22}-(g_{o2}\, g_{o2})/g_{oo}= \rho^2/(1 - \chi^2)$ and $\gamma_{33} = g_{33}-(g_{o3}\, g_{o3})/g_{oo} = 1$. So,

$$d\ell^2 = \gamma_{11}dx^1dx^1 + \gamma_{22}dx^2dx^2 + \gamma_{33}dx^3dx^3 = d\rho^2 + dz^2 + \gamma_{22}\,d\varphi^2.$$

Finally, we can write

$$d\ell^2 = d\rho^2 + dz^2 + \rho^2 d\varphi^2/(1 - \omega^2\rho^2/c^2) \qquad (8.b.3).$$

From (Bb.3) we see that the "real" circumference of a circle with radius ρ is

$$\ell = 2\pi\rho/(1 - \omega^2\rho^2/c^2)^{1/2} > 2\pi\rho \qquad (8.b.4),$$

where $2\pi\rho$ is the circumference measured by one that is not in the rotating system. So, if one fixed on the Earth is measuring the circumference of the equator, putting $\rho \sim 6.300$ km, $c \sim 300.000$ km/s , $\omega = 2\pi/T$ and T = 24 h ~ 9 10^4s will verify that $\ell/2\pi\rho = 1/(1 - \omega^2\rho^2/c^2)^{1/2} \sim 1+10^{-13}$.

Appendix A. Curvature Tensor and Space Curvature.

In GR there are two important tensors, one of rank four $R^{\sigma}{}_{\lambda\nu\mu}$ called *Riemann-Christoffel* tensor and another of rank two $R_{\lambda\mu}$ named *Ricci* tensor. The first one is defined by

$$R^{\sigma}{}_{\lambda\nu\mu} = \partial(\Gamma_{\lambda\nu}{}^{\sigma})/\partial x^{\mu} - \partial(\Gamma_{\lambda\mu}{}^{\sigma})/\partial x^{\nu} + (\Gamma_{\lambda\nu}{}^{\tau})(\Gamma_{\mu\tau}{}^{\sigma}) - (\Gamma_{\lambda\nu}{}^{\tau})(\Gamma_{\tau\nu}{}^{\sigma}) \quad (A.1)$$

and the second one obtained by index contraction of (A.1):

$$R_{\lambda\mu} = \partial(\Gamma_{\lambda\sigma}{}^{\sigma})/\partial x^{\mu} - \partial(\Gamma_{\lambda\mu}{}^{\sigma})/\partial x^{\sigma} + (\Gamma_{\lambda\sigma}{}^{\tau})(\Gamma_{\mu\tau}{}^{\sigma}) - (\Gamma_{\lambda\nu}{}^{\tau})(\Gamma_{\tau\sigma}{}^{\sigma}) \quad (A.2).$$

The *Riemann-Christoffel* tensor (A.1) is known as the *curvature tensor* and the scalar

$$R^{\lambda\lambda} = g^{\lambda\sigma} R_{\sigma\lambda} \qquad (A.3)$$

is named *curvature invariant*. It measures a property of the Riemannian space that it is analogous to the curvature of a 2-dim surface described by (2.1.5) $ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$. Putting $\theta = x^1$ and $\varphi = x^2$ we verify that[3,10]

$$R^{\lambda\lambda} = - 2/a^2 \qquad (A.4).$$

As the *Gaussian curvature* [3,10] of a sphere of radius a is known to be $1/a^2$ we verify that (A.4) shows that curvature invariant $R^{\lambda\lambda}$ is proportional to the Gaussian curvature. Note that when the space dimension is larger than 2-dim the relationship of these tensors to the "curvature" of the space is even more remote.

In certain Riemannian spaces of n-dim it is possible to find a coordinate-system (X) covering all the points of the space, in terms of which the metric becomes

$$ds^2 = \sum_{\lambda} \varepsilon_{\lambda}(dX^{\lambda})^2 \qquad (A.5)$$

where the ε_{λ} are positive or negative *constants*. In these conditions one can show[3] that (A.1), (A.2) and (A.3) are equal to zero. Such spaces are *flat* by analogy with the Euclidean space whose metric is of the type (A.5) when Cartesian coordinates are used.

In a flat space one has necessarily $R^{\sigma}{}_{\lambda\nu\mu} = 0$. It is possible to show the converse; if $R^{\sigma}{}_{\lambda\nu\mu} = 0$ throughout the space, then the space is flat. In other words, the space is flat if the *curvature tensor* $R^{\sigma}{}_{\lambda\nu\mu}$ is zero.

Appendix B. Absolute Differentiation, Covariant Derivative and Geodesics.

(B.1) Absolute Differentiation.

The *absolute differentiation*[1-3,5,8,11] of a vector field A^μ, for instance, is denoted by the symbol D defined by the operation

$$DA^\mu = dA^\mu + \Gamma_{\lambda\nu}{}^\tau A^\lambda dx^\nu \qquad (B.1.1),$$

where d is the *ordinary differentiation*.

(B.2) Covariant Derivative.

The *covariant derivative* along a curve is defined by

$$DA^\mu/D\tau \equiv A^\mu{}_{;\nu} (dx^\nu/d\tau) = dA^\mu/d\tau + \Gamma_{\lambda\nu}{}^\tau A^\lambda (dx^\nu/d\tau) \qquad (B.2.1),$$

which is a vector because it is a product of the tensor $A^\mu{}_{;\nu}$ and the vector $(dx^\nu/d\tau)$. Note that $dA^\mu/d\tau$ is not a vector.[2]

Along a geodesic the velocity field $v^\mu = dx^\mu/d\tau$ which is tangent to the curve in each point obeys the equation

$d^2x^\mu/d\tau^2 + \Gamma_{\lambda\nu}{}^\tau (dx^\lambda/d\tau) (dx^\nu/d\tau) = 0$, that is,

$dv^\mu/d\tau + \Gamma_{\lambda\nu}{}^\tau v^\lambda v^\nu = 0$

This implies, using (B.2.1), that

$$Dv^\mu/d\tau = 0 \qquad (B.2.2),$$

that is, one verifies that along a geodesic v^μ = constant. Note that in a Riemannian space ("curve") as $\Gamma_{\lambda\nu}{}^\tau \neq 0$ a particle even in absence of a physical interaction has *inertial accelerations*[see Section 7].

For flat spaces found in Special Relativity(SR) a free particle, not submitted to external forces, moves along a geodesic ("straight line") with constant velocity v^μ and null acceleration d^2v^μ/dt^2. As a *null geodesic* in Minkowski Space (MS) is given by $ds^2 = (dx^4)^2 - (d\mathbf{r})^2 = 0$ we get $(d\mathbf{r}/\mathbf{dt})^2 = v^2 = c^2$, showing that it represents the history of a light ray motion.

In GR in an obvious generalization of the SR, it is postulated that a free particle describes a geodesic. The hypothesis that free particles describe geodesics is called *Geodesic Principle* and corresponds to an extension of the *Inertia Principle* of Galileo. In fact in GR the "free

particle" is immersed in a Riemannian geometry dictated by a gravitational field. The interaction is "substituted" by the geometry.

Appendix C. Spacetime Intervals and Vectors.

In Euclidean space the separation between *two points* is measured by the distance between two points. This distance is purely spatial and is always positive. In a Riemannian spacetime the separation between *two events* takes into account not only the spatial separation between the events but also their temporal separation. In this case, between two events, we have *spacelike intervals* that are defined taking into account the metric

$$ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu \qquad (C.1).$$

(a) $ds^2 < 0$ → **Timelike interval**.

In Minkowski spacetime we have $c^2\Delta t^2 > \Delta r^2$. For two events separated by a *timelike interval* there is enough time between them to exist a cause-effect relationship between them.

(b) $ds^2 > 0$ → **Spacelike interval**.

In Minkowski spacetime we have $c^2\Delta t^2 < \Delta r^2$. When a *spacelike interval* separates two events there is no enough time between their occurrences that can be explained as being created by a causal relationship crossing the spatial distance between the two events with the speed of light or slower.

(c) $ds = 0$ → **Lightlike interval or null interval.**

In Minkowski spacetime $c^2\Delta t^2 = \Delta r^2$. In *lightlike interval* the spatial distance between two events is exactly the time interval elapsed between the two events. Events connected by photons have *lightlike* or *null intervals.*

Vectors.

In Riemannian spacetime a vector q^λ is defined as *timelike, spacelike* or *null* if the following conditions are, respectively, obeyed[8]

$$g_{\lambda\nu}(x)\, q^\lambda q^\nu < 0\ , > 0 \ \text{ or } = 0.$$

Appendix D. Paths of particles moving in the vicinity of spherical massive objects using Lagrangian formalism.

As seen in Section (3.4) the path of massive particles and photons described, respectively, by timelike and null geodesics can be obtained using the Lagragian formalism. This is done solving the Euler-Lagrange equations

$$d(\partial L/\partial \dot{x}^{\mu})/ds - \partial L/\partial \dot{x}^{\mu} = 0 \qquad \text{(D.1)},$$

where $x^{\alpha} = x^{\alpha}(u)$, $\dot{x}^{\alpha} = dx^{\alpha}/ds$ and

$$L(\dot{x}^{\alpha}, x^{\alpha}) \equiv (1/2)g_{\mu\nu}(x^{\alpha})\,(dx^{\mu}/ds)\,(dx^{\nu}/ds) \qquad \text{(D.2)}.$$

Solving (D.1) let us see how to determine the motion of massive particles and of photons in the vicinity of a spherical massive object. Following Sec. 5 the metric for a spherical object with mass M is the SM ds^2 that in spherical polar coordinates (r, θ, φ) is given by

$$ds^2 = (1-2\chi/r)\,c^2dt^2 - dr^2/(1-2\chi/r) - r^2d\theta^2 - r^2\sin^2\theta d^2\varphi \qquad \text{(D.3)},$$

where $\chi = GM/c^2$, $x^o = t$, $x^1 = r$, $x^2 = \theta$ and $x^3 = \varphi$.

(D.a) Massive Particle.

Taking into account the metric (D.3) one can show that the Lagrangian (D.2) becomes written as[8]

$$L(\dot{x}^{\alpha}, x^{\alpha}) = (1/2)\{\, c^2(1-2\chi/r)(dt/ds)^2 - (1-2\chi/r)^{-1}(dr/ds)^2 - r^2[(d\theta/ds)^2 + (d\varphi/ds)^2\sin^2\theta\,\} \qquad \text{(D.a.1)}.$$

Because of the spherical symmetry, there is no loss of generality in confining our attention to particles moving in the "equatorial plane" with $\theta = \pi/2$. In this case (D.a.1) is simplified and for $\mu = 2$, that is, $x^2 = \theta$ we get from (D.1) and (D.a.1),

$$F(r)^{-1}(d^2r/ds^2) + (mc^2/r^2)(dt/ds)^2 - F(r)^{-2}(m/r^2)(dr/ds)^2 - r(d\varphi/ds)^2 = 0, \qquad \text{(D.a.2)}$$

where $F(r) = (1-2\chi/r)$.

Since t and φ are cyclic coordinates we get from (D.1), putting $t^+ = dt/d\tau$ and $\varphi^+ = d\varphi/d\tau$:

$$\partial L/\partial t^+ = \text{const} \quad \text{and} \quad \partial L/\partial \varphi^+ = \text{const} \qquad \text{(D.a.3)}.$$

With (D.a.3) and $\theta = \pi/2$ in (D.a.2) we obtain

$$(1-2\chi/r)\,t^+ = k \quad \text{and} \quad r^2\varphi^+ = h \qquad \text{(D.a.4)},$$

where k and h are integration constants. The first one gives the relation between the coordinate time t and the proper time τ and the second one is

analogous to the equation of angular motion conservation. We have from (D.3) with $\theta = \pi/2$:

$$c^2F(r)t^{+2} - F(r)^{-1}r^{+2} - r^2\,\varphi^{+2} = c^2 \qquad \text{(D.a.5)},$$

which is much less complicated than (D.5) and yields an equation analogous that expressing the energy conservation. Indeed, taking into account (D.a.4) we obtain, putting $u = 1/r$ and where $E = c^2(k^2-1)/h^2$:[8]

$$(du/d\varphi)^2 + u^2 = E + (2MG/h^2)u + (2GM/c^2)u^3 \qquad \text{(D.a.6)}.$$

The analogous Newtonian equation which is given by

$$(du/d\varphi)^2 + u^2 = E + (2MG/h^2)u \qquad \text{(D.a.7)},$$

where $h = r^2(d\varphi/dt)$ is the angular momentum per unit of mass has a well known solution [8,9]

$$u = 1/r = (GM/h^2)[1 + e\cos(\varphi - \varphi_o)].$$

The general relativistic effect introduced by the cubic term u^3 in planetary motion in (D.a.6) is analyzed in reference 8 (pag.144-146).

Vertical Free-Fall.

In this case φ = constant, thus from (D.a.6) $\rightarrow \varphi^+ = 0 \rightarrow h = 0$. In this way putting $\varphi^+ = 0$ and $(1-2\chi/r)\,t^+ = k$ in (D.a.5) we obtain,

$$r^{+2} - ck^2 + c^2(1-2\chi/r) = 0 \qquad \text{(D.a.8)}.$$

If the particle is at rest ($r^+ = 0$) when $r = r_o \rightarrow k^2 = 1 - 2\chi/r_o$, showing that k is not an universal constant, but depends on r_o, that is, on the geodesic. In particular, if $r^+ \rightarrow 0$ when $r \rightarrow \infty$, then $k = 1$.

Differentiating (D.a.8) we get,

$$r^{++} + GM/r^2 = 0 \qquad \text{(D.a.9)}.$$

This equation has the same form ($d^2r/dt^2 + GM/r^2 = 0$) as that found in the Newtonian approach; remembering that in (D.a.9) the coordinate **r is not the vertical distance** and the derivative **r^{++}** is with respect of the **proper time** τ of the particle in motion, not the universal time t.

Putting $k^2 = 1 - 2\chi/r_o$ in (D.a.8) we obtain

$$r^{+2}/2 = MG(1/r - 1/r_o) \qquad \text{(D.a.10)}$$

which is "similar" to that in the Newtonian theory. Integrating (D.a.10) assuming that r_o is the initial coordinate of particle at $\tau = 0$ we can calculate the proper time τ experienced by the particle in falling from r_o up to r :

$$t = (1/\sqrt{2GM})\int_r^{r_0}[r_0 r/(r_0 - r)]^{1/2}dr \qquad \text{(D.a.11)}.$$

It permit us to establish the relation $r = r(\tau)$. To obtain $r = r(t)$ we take into account $dt/dr = (dt/d\tau)/(d\tau/dt)$ with (D.a.4),

$$dt/d\tau = k/(1 - 2\chi/r) = (1-2\chi/r_o)^{1/2}/(1 - 2\chi/r).$$

and with (D.a.10) getting, with $\chi = GM/c^2$:

$$v(t) \equiv dr/dt = -[r^{3/2}(r_o - 2m)^{1/2}]/\{c(2m)^{1/2}\,(r - 2m)(r_o - r)^{1/2}\} \qquad \text{(D.a.12)}$$

The way in which the coordinate time t depends on r for a radially falling particle becomes more comprehensible if, considering for simplicity that it is at rest at infinity $r_o \to \infty$, we compare its *coordinate speed,* obtained from (D.a.12),

$$v(r) = |dr/dt| = (2mc^2)^{1/2}(r - 2m)/r^{3/2}$$

with the classical Newtonian speed[9]

$$v_c(r) = \sqrt{2mc^2}/r^{1/2}.$$

This difference is clearly seen in Figure (D.1).

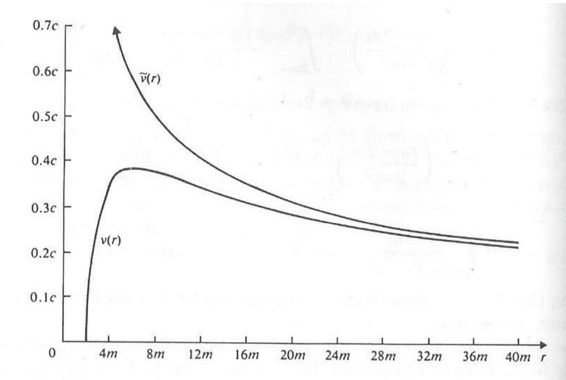

Fig.(D.1). Comparison of the coordinate speed v(r) with the classical Newtonian speed $v_c(r)$ for a particle falling from rest at infinite.

Motion in a circle.[8]

For a circular motion in the equatorial plane we have r = const and, consequently, $r^{+} = r^{++} = 0$. Equation (D.a.2) then reduces to

$$mc^2 t^{+2} = r^3\,\varphi^{+2} \qquad \text{giving} \qquad (d\varphi/dt)^2 = GM/r^3 \quad \text{(D.a.13)}$$

From (D.a.13) we see that for $\Delta\varphi = 2\pi$ the coordinate time Δt for a complete revolution is given by

$$\Delta t = 2\pi(r^3/GM)^{1/2} \qquad \text{(D.a.14).}$$

This expression is exactly the same as the Newtonian expression for the period of a circular orbit of radius r, that is, Kepler´s third law. However, in the relativist case we cannot say that r is the radius of the orbit, but we see that the spatial distance traveled in one complete revolution is 2πr, just as in the Newtonian case.

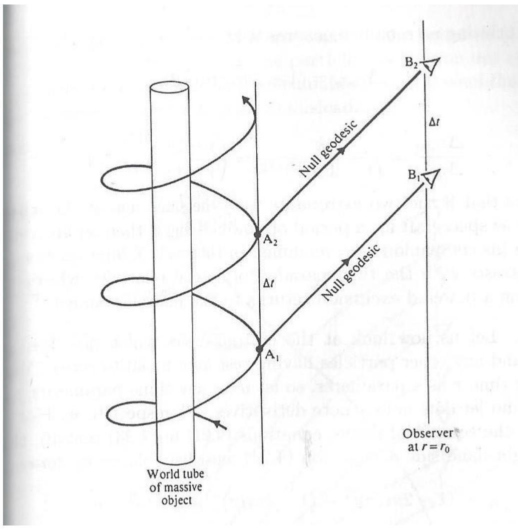

Fig.(D.1). Spacetime diagram shown a circular orbit as viewef by a fixed observer at $r = r_{o.}$

Fig.(D.1) is a space time diagram illustrating one complete revolution as viewed by an observer fixed at a point $r = r_o$. B_1 is the event observed when the particle is at the start point of orbit at A_1 , while B_2 is that of his viewing its completion at A_2. If the coordinate time difference between A_1 and A_2 is Δt the coordinate time difference between B1 and B2 is also Δt , according to (5.3.14). So, the proper time $\Delta\tau_o$ which the observer measures for the orbital period Δt is given by

$$\Delta\tau_o = (1-2\chi /r_o)^{1/2}\,\Delta t \qquad \text{(D.a.15).}$$

As $r_o \rightarrow\infty$, $\Delta\tau_o \rightarrow\infty$, so Δt is the orbital period as measured by an observer at infinity. So, as Δt is directly observed, according to (D.a.14) the coordinate r also *could be measured* if the mass M is known.

Now, let us determine the proper lapse of time measured by an observer moving with the rotating particle A. In this case the relationship between t and τ is obtained using (D.a.4) and (D.a.13). So,

$$t^{+2} = k^2r^2/(r - 2\chi)^2 \qquad \textbf{[a]} \qquad \text{and}$$

$$\varphi^{+2} = mc^2t^{+2}/r^3 = mc^2k^2/[r\,(r-2\chi)^2] \quad \textbf{[b]} \qquad \text{(D.a.16).}$$

Substituting (D.a.16) **[a]** in (D.a.5) and putting $r^+ = 0$, since now the observer moves in a circle, we obtain

$$k^2 = (r-2\chi)^2/[r(r - 3\chi)] \qquad \text{(D.a.18).}$$

Putting this k^2 in $(1-2\chi /r)\, t^+ = k$, given by (D.a.4), we get, as $t^+ = dt/d\tau$:

$$\Delta\tau = (1-2\chi/r)\, k^{-1}\Delta t = [(r-3\chi)/r]^{1/2}\, \Delta t = 2\pi\{(r^3/MG)\,(1-3MG/rc^2)\}^{1/2} \text{ (D.a.19).}$$

This equation shows that we can have circular orbits *only* for $k^2 > 0$, that is, only when $r > 3\chi = 3GM/c^2$. In the limit $r \rightarrow 3\chi$, $\Delta\tau \rightarrow 0$, *suggesting* that photons can orbit at $r = 3\chi$ .This indeed occurs(see reference[8]).

Appendix E. Length and Time in Schwarzschild Spacetime(SST).

The SST has the line element given by, with $\chi = 2M/c^2$:

$$ds^2 = c^2 d\tau^2 = (1-2\chi/r)\, c^2 dt^2 - dr^2/(1-2\chi/r) - r^2 d\theta^2 - r^2 \sin^2\theta d^2\varphi \qquad (E.1).$$

If we take a *slice* given by t = constant we obtain a 3-dim manifold. Putting the line element

$$ds^2 = g^*_{ij} dx^i dx^j \quad (i,j = 1,2,3,\ x^1 = r,\ x^2 = \theta,\ x^3 = \varphi),\quad g^*_{ij} = -g_{ij} \quad (E.2),$$

describing a *slice* which is a *space* rather than a spacetime. As g^*_{ij} is time independent we can observe two events at the same point in space occurring at different times. When M = 0 the line element (E.1) describes a *flat* spacetime, while the line element (E.2) describes the Euclidean *flat space* in spherical coordinates according to (2.1.4).

What about radial distances given by θ and φ constants? The line element (E.1) shows that for these infinitesimal radial distance dR is given by $dR \equiv dr/(1-2\chi/r)^{1/2}$, so dR > dr and r **no longer measures radial distances**. **Not even the origin** of the r coordinates **can be defined!**

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